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On perturbation methods in nonlinear stability theory

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This paper reconsiders formal expansion methods for the analysis of the nonlinear properties of a modal disturbance. A survey is given of the various types of expansions based on different assumptions, and their range and shortcomings are discussed. By introducing a well-defined amplitude, Watson's expansion in a time-dependent amplitude is developed into a rational method for uniquely determining Landau constants of arbitrary order. Complementary to the common orthogonality condition for points at the neutral curve, an alternative definition of the Landau constants is given for points in the unstable domain. The method is not restricted to small amplification rates but is invalid in the stable domain. The method of Reynolds and Potter for a direct attack on equilibrium states is extended into a class of rational methods. The methods in this class agree to within a rearrangement of the infinite expansion series but differ in their respective range of validity.

1. Introduction

Analytical perturbation methods are familiar and powerful approximation methods for solving nonlinear equations. These techniques represent the solution $f = f(x, \epsilon)$ by an expansion

$$f(x,\epsilon) = \sum_{m=0}^{\infty} f_m(x)\,\epsilon^m = f_0(x) + \epsilon f_1(x) + \dots \tag{1.1}$$

in terms of some parameter (or variable) ϵ that appears naturally or artificially in the problem. After finding $f_0(x)$ as the solution of the equations for $\epsilon = 0$, the functions $f_m(x)$ are usually governed by a sequence of simpler linear equations, which can be solved successively. In contrast to purely numerical approaches, the perturbation method provides the solution for some range of ϵ rather than a single value of the parameter, and the series representation often enhances the insight into the analytical structure of the solution.

Normally, the series solution (1.1) is truncated at rather low order in the small parameter ϵ because the handling of the rapidly increasing number of terms is laborious and susceptible to errors. This may give reliable results on tendencies of certain functions such as the gradient $\partial f/\partial\epsilon$ close to $\epsilon = 0$. However, the utility of the truncated perturbation series for actually representing the true solution at some finite value of ϵ cannot be assessed without at least rough information on the analytical properties of the solution and on the convergence of the series (1.1). Merely including higher-order terms may improve the approximation, but may on the other hand be useless if the series diverges in the relevant range of ϵ. Nevertheless, the information concealed in the higher-order terms can be profitable. Beyond the illustrative comparison of different approximations there are various more sophisticated techniques available (e.g. Shanks 1955; Gaunt & Guttmann 1974; Van

Dyke 1974) for extracting from these higher-order terms the necessary information on the analytical structure and recasting the series in some more suitable form. These techniques simply exploit the fact that the nature of the true solution is imprinted on the coefficients f_m even if the series diverges or poorly converges.

A first obvious yet not always satisfied requirement for these techniques is a rational approximation method, i.e. a consistent formal expansion procedure that can be carried on to arbitrary order. Secondly, this procedure must be prepared for automatic execution in order to take advantage of the capability of computers to handle large data sets and to repeat correctly the involved induction process. In this case, only a few terms need to be formally checked by hand. The lengthy equations for higher-order terms never appear explicitly but are internally generated. Finally, the set-up of the internal equations needs to be interlaced with suitable numerical methods for converting these into algebraic equations and for obtaining their solution which must keep step with the induction process.

A long list of references on applications of straightforward extended perturbation series in fluid mechanics and a survey of results for a variety of problems attacked at Stanford University has been given by Van Dyke (1975). His paper also elucidates some computational aspects of series extension and gives useful advice for programming. For some prototype series, Van Dyke (1974) has also discussed how the range of applicability can be extended, or the accuracy of the results increased, by analysing the coefficients and then recasting the series in another form. After additional experience with other problems, Van Dyke (1978) states that the three-step scheme of extension, analysis and improvement of perturbation series is 'an attractive alternative to finite difference computation, at least in simple problems'. The aim of the present investigation is to show that it may be also an attractive tool for the analysis of nonlinear stability of flows, which cannot be considered as a simple problem. This paper, however, concentrates only on the first basic requirement and considers the formal aspects of rational approximation methods for the most important classes of problems.

The essential steps in the study of nonlinear processes in flow stability, such as growth and equilibration of normal modes, mode selection, wavenumber selection or secondary instability, were discussed in detail by Stuart (1971), Busse (1978) and others. These surveys also indicate that much of our present understanding of nonlinear effects relies on perturbation methods or other semi-analytical approximations. Numerical analysis by directly solving the Navier–Stokes equations increasingly contributes in some specific cases like plane Poiseuille flow (Zahn *et al.* 1974; Herbert 1977, 1978*a*; Orszag & Kells 1980; Orszag & Patera 1981; Kleiser 1982). But even with advanced computers the systematic investigation of many important phenomena exceeds the limits of feasibility. Moreover, concerns persist about resolution of proper scales, numerical stability and the role of artificially introduced boundaries. The numerical experiments of Fasel (1974) clearly indicated that insufficient resolution in space or time may falsify the solution completely and it will certainly suppress possible relevant small-scale phenomena. Besides this, the basic mechanisms at work are often difficult to retrieve from the mass of numerical data obtained for a few points in parameter space.

On the other hand, rational perturbation methods are as yet only available for a special class of nonlinear stability problems that are related to the equilibrium states as they are observed as steady Taylor vortices or convection cells. For a wide class of basically unsteady problems most relevant to parallel flows, the methods currently in use suffer from unintelligible restrictions and provide no basis for a rational

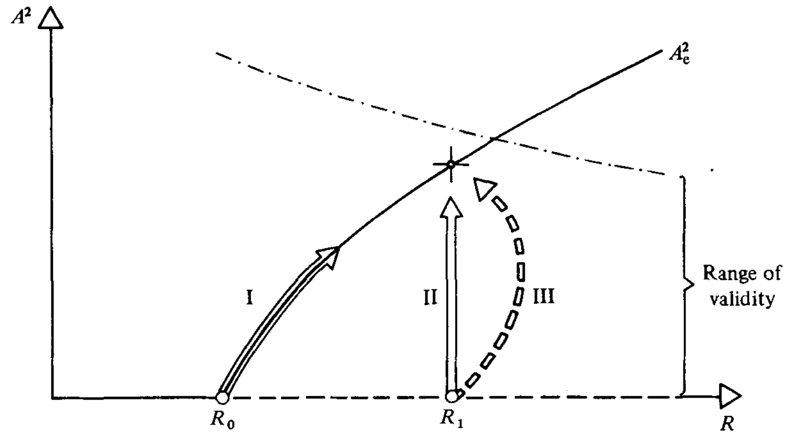

FIGURE 1. Bifurcation diagram for supercritical stability at wavenumber α_0 and illustration of (I) parameter expansion, (II) expansion in the amplitude $A(t)$, and (III) method of false problems. The range of validity may be different for each method.

approximation. Although asymptotic theories turned out to be very fertile at lowest order (Stewartson 1975), they have undergone little further development towards a quantitative analysis of nonlinear phenomena.

Historically, a variety of perturbation methods for the analysis of nonlinear stability has been independently developed in the fields of thermal convection and parallel flows. The only common feature is the use of some solution of the linear stability problem as a zeroth approximation, which is automatically selected by the choice of an expansion parameter appropriate for the problem under consideration.

In thermal convection, interest is primarily centred on the steady equilibrium states bifurcating from the basic state of resting fluid with pure heat conduction. Therefore it is natural to start from the steady equations of motion and to use, for a given wavenumber α_0, the solution at the point α_0, R_0 of the neutral curve (usually the critical point α_c, R_c) as zeroth approximation. The parameter ϵ then measures the distance from this point in terms of the equilibrium amplitude A_e or the difference $R-R_0$ in the relevant dimensionless parameter R, here the Rayleigh number. Given the bifurcation diagram at $\alpha = \alpha_0$ in figure 1, this parameter expansion follows the arrow labelled I along the curve $A_e^2(R-R_0)$ as ϵ increases. Expansions of this type have been introduced by Gorkov (1957) and Malkus & Veronis (1958). For two-dimensional convection between stress-free boundaries, Malkus & Veronis carried the analysis to sixth order, Kuo & Platzman (1961) with a modified, more elegant formulation to eighth order. This is an exceptional case, since the solution can be readily expressed by trigonometric functions, and it seems to be the only case where a clear indication of the restricted convergence domain is available. This restriction was partly overcome by Kuo (1961), who obtained a more rapidly converging series by expanding in a slightly different parameter.

Meanwhile, parameter expansions about a neutral solution were successfully applied for studies on finite-amplitude Taylor vortices in supercritical circular Couette flow (Reynolds & Potter 1967*a*; Kirchgässner & Sorger 1969). The method was also extended to the construction of time-periodic solutions bifurcating from plane Poiseuille flow (Joseph & Sattinger 1972; Chen & Joseph 1973). With the bifurcation diagram at α_0 shown in figure 2, the parameter expansion along arrow

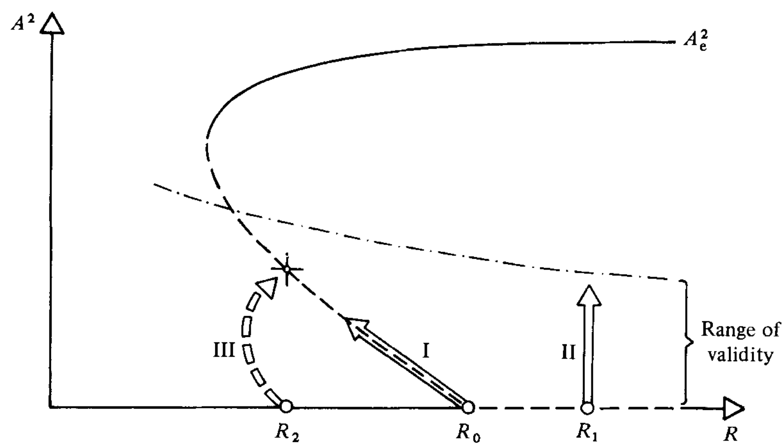

FIGURE 2. Bifurcation diagram for subcritical instability at wavenumber α_0 and illustration of (I) parameter expansion, (II) expansion in the amplitude $A(t)$, and (III) method of false problems. The range of validity may be different for each method.

I provides the threshold amplitudes for subcritical instability. A clear advantage of these parameter expansions is to provide at once the equilibrium solution for some range of $R-R_0$ instead of a single value.

Obviously, these otherwise successful methods are by nature not applicable to flows like pipe flow, which seem to be stable according to linear stability theory and exhibit no bifurcation point at finite R. Moreover, they are not applicable to processes involving the nonlinear growth or decay of disturbances. From a different point of view, one can also study the steady equilibrium state in figure 1 at some supercritical point α_0, $R_1 > R_0$ as it originates asymptotically from nonlinear growth of an unstable normal mode. For this second type of expansion along arrow II it is natural to use a coordinate expansion in the time-dependent amplitude $A(t)$. The appropriate zeroth approximation for $A \to 0$ is then given by the solution of the linear stability problem for the point α_0, R_1. Studies on unsteady processes are considered most important for unstable flows like plane Poiseuille flow where disturbance growth leads to breakdown into the turbulent motion. The aim is then to trace the nonlinear evolution of a normal mode at α_0, $R_1 > R_0$ for some finite time,† until other physical phenomena occur or the amplitude exceeds the range of validity, as indicated by arrow II in figure 2.

This second approach to nonlinear stability problems was suggested by Stuart (1960) for plane Poiseuille flow. Although an expansion in $A(t)$ was independently introduced by Palm (1960) in thermal convection, it was Stuart's work that opened the way for the development of formal methods for the analysis of modal disturbances by Watson (1960), Eckhaus (1965), Itoh (1974, 1977*a*) and others, as well as for a whole series of applications with fundamental results, e.g. Davey (1962) for circular Couette flow and Reynolds & Potter (1967*b*), Pekeris & Shkoller (1967) for plane

† Although normal modes can be considered as growing spatially as well, we restrict our attention to the classical case of temporal growth. Most of our arguments need only slight modification for the other case.

Poiseuille flow. One of the major achievements of these methods is the derivation of the Landau equation

$$\frac{1}{A}\frac{dA}{dt} = a_0 + a_1 A^2 + \ldots = a(A), \tag{1.2}$$

for the nonlinear amplification rate a as a function of the amplitude A (we consider all quantities in (1.2) as real). This equation is considered as representing the essence of the nonlinear disturbance behaviour. Consequently, the spatial structure of the disturbance can be disregarded and the analysis aims at determining the Landau constants $a_1, \ldots$ in the series (1.2), whereas a_0 is the linear amplification rate.

It is obvious that this class of asymptotic theories can be expected to be valid only for sufficiently small amplitudes $A(t)$. In spite of modifications and extensions, however, the range of applicability is restricted for other reasons. A problem of non-uniqueness occurs in determining the higher-order Landau constants beyond a_1, restricting most of the applications to the lowest-order approximation. Eagles (1971) found by rational arguments an additional condition to select a specific value of a_2, but a generic procedure was not revealed. Even at lowest order, application of the methods is restricted to the vicinity of the neutral curve, $|a_0| \ll 1$. As a consequence, various possibly valuable quantitative results for $a_0 \neq 0$, e.g. those of Eagles (1971), are subject to reservations. Important phenomena at small amplitude such as the interaction of different modes (Stuart 1962) are inaccessible to these techniques, since not all Landau constants in the coupled amplitude equations can be determined and the condition on the amplification rates cannot be satisfied. Moreover, Davey & Nguyen (1971) found that the expansion in a single amplitude $A(t)$ is invalid in the stable domain, $a_0 < 0$, owing to possible resonance with mean-flow modes.

For a direct attack on equilibrium states, a 'method of false problems' was first suggested by Reynolds & Potter (1967*b*). These methods rest on an expansion in $A(t)$, but the equilibrium condition $dA/dt = 0$ for $A = A_e$ is exploited in the derivation of the equations. Hence non-physical solutions of false problems are obtained for $A \neq A_e$. For sufficiently small equilibrium amplitudes, the methods can be applied for arbitrary points in the stable or unstable domain, as indicated by the broken arrows labeled III in figures 1 and 2. It is an obvious disadvantage of these methods to provide the equilibrium solution only for a single point, whereas expansions of type I provide this solution for some range of the perturbation parameter. However, methods of false problems are the only ones that can be applied to problems without a neutral curve.

The results of Itoh (1977*b*) for the centre mode in pipe flow indicate that Reynolds & Potter's method can be invalidated by resonance with the harmonic equations. Itoh (1977*a*) suggested a modified method for determining a_1 similar to that used by Ellingsen, Gjevik & Palm (1970). The relation between Reynolds & Potter's method and Itoh's method was discussed by Davey (1978), who pointed out that the methods differ only by a rearrangement of the terms of an infinite series. Davey also emphasized the importance of higher-order terms for conclusive results. These higher-order terms, however, suffer from the same non-uniqueness as in Watson's method. The introduction of arbitrary conditions (Coffee 1977) is inappropriate in overcoming this problem and leaves the values of the Landau constants inconclusive. Herbert (1978*b*, 1980) introduced a well-defined amplitude in Reynolds & Potter's method and found all higher-order terms uniquely determined. Landau constants up to a_7 were calculated for equilibrium states in plane Poiseuille flow in order to shed some light on the convergence of Landau's series. As a side result, the use of the same amplitude definition with Watson's method provided conditions for determining the constants irrespective of the vicinity of the neutral curve.

The aim of the present paper is primarily to present in §2 a consistent formulation of a rational expansion in the time-dependent amplitude $A(t)$, with special emphasis on the fundamental assumptions and the validity of the expansion series. We restrict our discussion to the simplest case of a single mode, although our alternative definition of the Landau constant provides a new rational basis for studies on interacting modes. We also disregard the interesting field of wave packets. In §3 we compare our formal results with those of previous work and shed some light on the controversial discussion of amplitude expansions. In §4 we suggest a class of rational methods of false problems which contains the methods of Reynolds & Potter and Itoh as special cases.

2. Expansion formalism for growing disturbances

2.1. *Fourier analysis of the basic equations*

We consider the two-dimensional flow of an incompressible fluid of viscosity ν and density ρ between parallel planes of distance $2h$, which is driven by a constant (mean) pressure gradient. The Navier–Stokes equations may be written in the form

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{\partial p}{\partial x}+\frac{1}{R}\Delta u, \tag{2.1}$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{\partial p}{\partial y}+\frac{1}{R}\Delta v, \tag{2.2}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{2.3}$$

where $\Delta=\partial^2/\partial x^2+\partial^2/\partial y^2$, x denotes the direction of the pressure gradient parallel to the planes, y the distance normal to them measured from the channel centre, u, v the corresponding velocity components, p the pressure, t the time and R the Reynolds number. All quantities have been made non-dimensional with respect to the channel half-width h, the mid-channel velocity U_0 in steady flow, and the reference pressure ρU_0^2. The Reynolds number is defined by $R=U_0h/\nu$. The boundary conditions require that both velocity components vanish at the walls:

$$u(x,y,t)=v(x,y,t)=0 \quad \text{at} \quad y=\pm 1. \tag{2.4}$$

The basic laminar flow is given by the plane Poiseuille flow

$$U(y)=1-y^2, \quad V=0, \quad P=-\frac{2}{R}x, \tag{2.5}$$

which solves the steady version of (2.1)–(2.4) for all Reynolds numbers.

To satisfy the continuity equation (2.3) we express the velocity field by a stream function ψ such that

$$u=\frac{\partial \psi}{\partial y}, \quad v=-\frac{\partial \psi}{\partial x}, \tag{2.6}$$

and the Navier–Stokes equations become

$$\frac{\partial \Delta\psi}{\partial t}+\frac{\partial \psi}{\partial y}\frac{\partial \Delta\psi}{\partial x}-\frac{\partial \psi}{\partial x}\frac{\partial \Delta\psi}{\partial y}=\frac{1}{R}\Delta\Delta\psi. \tag{2.7}$$

The stream function $\hat{\psi}(x,y,t)$ of an unsteady disturbance added to the basic flow is then governed by the equation

$$\left[\frac{1}{R}\Delta\Delta-\left(U\frac{\partial}{\partial x}\Delta-\frac{d^2U}{dy^2}\frac{\partial}{\partial x}\right)-\frac{\partial}{\partial t}\Delta\right]\hat{\psi}=\left(\frac{\partial\hat{\psi}}{\partial y}\frac{\partial}{\partial x}-\frac{\partial\hat{\psi}}{\partial x}\frac{\partial}{\partial y}\right)\Delta\hat{\psi}, \tag{2.8}$$

with boundary conditions

$$\frac{\partial\hat{\psi}}{\partial y}=\frac{\partial\hat{\psi}}{\partial x}=0 \quad \text{at} \quad y=\pm 1. \tag{2.9}$$

In the linear stability theory, the right-hand side of (2.8) is set equal to zero, and a solution of (2.8) and (2.9) is found in the form

$$\hat{\psi}(x,y,t)=e^{a_0 t}\phi_{10}(y)\,e^{i(\alpha x-\omega_0 t)}, \tag{2.10}$$

where α is the wavenumber, a_0 the amplification rate and ω_0 the frequency. For given R and α, $\lambda_0=a_0-i\omega_0$ is a complex eigenvalue of the Orr–Sommerfeld problem and $\phi_{10}(y)$ the related eigenfunction subject to some normalization. For plane Poiseuille flow, interest is centred on the symmetric eigenfunction related to the principal eigenvalue λ_0. In this case, the channel centre $y_0=0$ is a suitable location for the normalization

$$\phi_{10}(y_0)=1 \quad \text{at} \quad y_0=0, \tag{2.11}$$

which fixes amplitude and phase of the wave (2.10). The regions of stability ($a_0<0$) and instability ($a_0>0$) in the (R,α)-plane are separated by the neutral curve $a_0(\alpha,R)=0$.

If nonlinear terms are taken into account, the disturbance reacts with itself, with its complex conjugate and with the mean flow – which results in the generation of harmonics, a mean-flow distortion and a distortion of the fundamental, respectively. Moreover, the frequency and amplification rate will change with the finite size of the disturbance. Therefore it seems natural to represent the nonlinear disturbance as the Fourier series

$$\hat{\psi}(x,y,t)=\sum_{n=-\infty}^{\infty}\psi_n(y,t)\,e^{in\theta}, \quad \theta=\alpha x-\gamma(t). \tag{2.12}$$

With real $\gamma(t)$ any growth of the disturbance is absorbed into the Fourier coefficients ψ_n. For a real solution,

$$\psi_{-n}(y,t)=\tilde{\psi}_n(y,t) \tag{2.13}$$

must be satisfied, where the tilde denotes the complex conjugate. By using the series (2.12) we restrict the class of solutions to those that are periodic in x with wavenumber α. Moreover, we imply the strong assumption that the nonlinear solution is uniquely determined by the fundamental component. The validity of this assumption remains to be discussed. Substituting $\hat{\psi}$ according to (2.12) into (2.8) and (2.9) and separating out the coefficients of like exponentials exp $(in\theta)$ provides an infinite set of equations and boundary conditions for the Fourier components ψ_n:

$$\left\{L_n-\left(\frac{\partial}{\partial t}-in\frac{d\gamma}{dt}\right)M_n\right\}\psi_n=\sum_{\nu=0}^{n}N[\psi_\nu,\psi_{n-\nu}]+\sum_{\nu=1}^{\infty}\{N[\psi_{-\nu},\psi_{n+\nu}]+N[\psi_{n+\nu},\psi_{-\nu}]\}, \tag{2.14}$$

$$\frac{\psi_n}{\partial y}=n\psi_n=0 \quad \text{at} \quad y=\pm 1, \tag{2.15}$$

where the operators L_n, M_n, N are

$$L_n = \left(\frac{\partial^2}{\partial y^2} - n^2\alpha^2\right)^2 - in\alpha R\left[U(y)\left(\frac{\partial^2}{\partial y^2} - n^2\alpha^2\right) - \frac{d^2U(y)}{dy^2}\right],$$

$$M_n = R\left(\frac{\partial^2}{\partial y^2} - n^2\alpha^2\right),$$

$$N[\psi_k, \psi_l] = i\alpha R\left(l\frac{\partial \psi_k}{\partial y} - k\psi_k\frac{\partial}{\partial y}\right)\left(\frac{\partial^2}{\partial y^2} - l^2\alpha^2\right)\psi_l.$$

It is obvious from (2.15) for $n = 0$ that only two boundary conditions are available for the solution ψ_0 of the fourth-order differential equation (2.14). The first degree of freedom is due to the fact that the stream function is only determined to within an additive constant that is irrelevant for the physical solution in terms of the velocity components (2.6). Integrating the equation for ψ_0 once over y introduces a time-dependent constant of integration. By comparison with the mean of (2.1) it is easily seen that this second degree of freedom is equivalent to a change with time of the mean-pressure gradient, where the mean is taken with respect to x over one wavelength $2\pi/\alpha$. We set this constant of integration equal to zero according to our assumption that the mean-pressure gradient should be constant. The flow rate through the channel then will change with the size of the disturbance. Therefore the assumption of a constant mean-pressure gradient is valid only for temporally growing disturbances. As an alternative that is valid for both, temporal or spatial growth, one could determine the constant of integration such that a constant mass flux through the channel is maintained, with an associated variation of the mean-pressure gradient with the size of the disturbance. In either way we obtain an equation for unique determination of $u_0 = \partial\psi_0/\partial y$.

2.2. *Expansion in the amplitude*

The system (2.14) of coupled nonlinear partial differential equations is very difficult to solve, and we therefore seek a solution by a perturbation method expanding about the solution (2.10) of the linear stability problem at fixed R and α. Since no small parameter appears naturally in this problem, we introduce artificially some suitable measure for the size of the disturbance stream function $\hat{\psi}$. According to our assumption that $\hat{\psi}$ is completely determined by the fundamental ψ_1 we measure this size by the amplitude $A(t)$ of the fundamental. The initially exponential growth of $A(t)$ according to the linear theory will be modified as nonlinearity becomes significant and probably $A(t)$ approaches an equilibrium value. Even if $A(t)$ is bounded, however, it is questionable whether the amplitudes at large times will be in the range of validity of the perturbation expansion. Therefore, we restrict our attention to finite but sufficiently small amplitudes, which may be reached at some finite time.

The definition of the amplitude as an unambiguous measure for the size of the fundamental is crucial for the expansion procedure. Formally, the fundamental can be written as

$$\psi_1(y,t) = A(t)\,\phi_1(y,t). \tag{2.16}$$

Since we require that $A(t)$ comprehends any linear or nonlinear variation in the size of ψ_1, the size of the function $\phi_1(y,t)$ must be constant, independent of time. The problem of measuring the size of $\phi_1(y,t)$ bears similar arbitrariness as that of normalizing the eigenfunction $\phi_1(y)$ in (2.10). In a mathematically rigorous manner, some norm could be chosen, e.g. the maximum norm or a norm based on a scalar product involving an integral across the channel (Chen & Joseph 1973). For computational simplicity, however, the size is usually measured at some suitable fixed y-position as in (2.11). For the principal eigenmode in plane Poiseuille flow it happens that $|\phi_{10}(y)|$ assumes a maximum at the channel centre $y_0 = 0$. For ψ_1, or ϕ_1, this property can be expected only for sufficiently small amplitudes. Strictly speaking, strong nonlinear distortion could result in ψ_1 vanishing at y_0 without being zero everywhere across the channel, and hence could lead to a failure of the local measure. Nevertheless, we adopt a local measure at the channel centre for an easier comparison with previous work. Introduction of some other measure will not affect the essence of our formal expansion.

For the nonlinear solution associated with the symmetric principal eigenmode in plane Poiseuille flow we define the (real) amplitude $A(t)$ by

$$A(t) = |\psi_1(y_0, t)|, \quad y_0 = 0. \tag{2.17}$$

From (2.16) we obtain $|\phi_1(y_0, t)| = 1$. In order to fix the as-yet arbitrary phase of the disturbance we set

$$\phi_1(y_0, t) = 1, \quad y_0 = 0. \tag{2.18}$$

As the amplitude tends to zero, the $O(A)$ terms of the nonlinear solution must represent the solution (2.10) of the linear stability problem. Hence at this order we require

$$\frac{1}{A}\frac{dA}{dt} \to a_0, \quad \frac{d\gamma}{dt} \to \omega_0, \quad \phi_1(y, t) \to \phi_{10}(y) \quad \text{as} \quad A \to 0, \tag{2.19}$$

while the forced components ψ_n, $n \neq 1$, tend to zero more rapidly.

From substituting (2.16) into the forcing terms on the right-hand side of (2.14) it is obvious that the leading term of ψ_2 is $O(A^2)$ and is produced by the first sum. By induction, this first sum generates all higher harmonics ψ_n that are $O(A^n)$, but it does not contribute to the leading term of ψ_0. Complying with (2.13), these estimates suggest that we seek a solution in the form

$$\psi_n(y, t) = A^{|n|}\phi_n(y, t), \tag{2.20}$$

where $\phi_n = O(1)$, $n \neq 0$, and $\phi_0 = O(A^2)$ as $A \to 0$. The exceptional role of ϕ_0 is due to the fact that the $O(1)$ terms of the total stream function ψ represent the basic flow which is split off from $\hat{\psi}$. The property (2.20) of the ψ_n to contain no terms of an order smaller than $O(A^n)$ brings about the desired decoupling of the nonlinear equations. Substituting (2.20) into (2.14) and (2.15) and equating like powers of $A(t)$ we obtain an infinite set of equations for the ϕ_n:

$$\left\{L_n - \left(\frac{\partial}{\partial t} + n\lambda\right) M_n\right\} \phi_n = F_n, \tag{2.21}$$

$$F_n = \sum_{\nu=0}^{n} N[\phi_\nu, \phi_{n-\nu}] + \sum_{\nu=1}^{\infty} A^{2\nu}\{N[\phi_{-\nu}, \phi_{n+\nu}] + N[\phi_{n+\nu}, \phi_{-\nu}]\},$$

$$\frac{\partial \phi_n}{\partial y} = n\phi_n = 0 \quad \text{at} \quad y = \pm 1, \tag{2.22}$$

where

$$\lambda = a - i\omega, \quad a = \frac{1}{A}\frac{dA}{dt}, \quad \omega = \frac{d\gamma}{dt}.$$

Since all ϕ_n are either $O(1)$ or $O(A^2)$ as $A \to 0$, the right-hand side of (2.21) can generate only higher-order terms in ascending powers of A^2. Hence the Poincaré stretching of the eigenvalue λ can be carried out in terms of A^2 rather than A. In this way, there is no need for first introducing odd powers of A and ultimately finding their coefficients equal to zero. It is consistent with (2.21) and (2.22) to seek a solution of the form

$$\phi_n(y,t) = \sum_{m=0} \phi_{nm}(y) A^{2m}, \quad \phi_{00} \equiv 0, \tag{2.23}$$

$$\lambda = \sum_{m=0}^{\infty} \lambda_m A^{2m}, \quad \lambda_m = a_m - i\omega_m. \tag{2.24}$$

Substituting into (2.21) and equating like powers of A^2, we obtain

$$\{L_n - (2ma_0 + n\lambda_0) M_n\}\phi_{nm} = \sum_{\mu=1}^{m} [2(m-\mu) a_\mu + n\lambda_\mu] M_n \phi_{nm-\mu} + F_{nm}, \tag{2.25a}$$

$$F_{nm}(y) = \sum_{\nu=0}^{n} \sum_{\mu=0}^{m} N[\phi_{\nu\mu}, \phi_{n-\nu m-\mu}] + \sum_{\nu=1}^{m} \sum_{\mu=0}^{m-\nu} \{N[\phi_{-\nu\mu}, \phi_{n+\nu m-\nu-\mu}] + N[\phi_{n+\nu m-\nu-\mu}, \phi_{-\nu\mu}]\}, \tag{2.25b}$$

where summations from 1 to m are omitted when $m = 0$. We retain the notation for the operators and understand that $\partial/\partial y$ is replaced by d/dy when applied to the functions $\phi_{nm}(y)$. The boundary conditions on ϕ_{nm} are

$$\frac{d\phi_{nm}}{dy} = n\phi_{nm} = 0 \quad \text{at} \quad y = \pm 1, \tag{2.26}$$

with the two degrees of freedom for ϕ_{0m} fixed as earlier discussed. In addition we have from (2.18) the infinite set of conditions

$$\phi_{10}(y_0) = 1, \quad \phi_{1m}(y_0) = 0, \quad m > 0 \quad \text{at} \quad y_0 = 0. \tag{2.27}$$

We turn now to show that the set of equations (2.25)–(2.27) determines uniquely the complex Landau constants λ_m and functions ϕ_{nm}, and hence the complete nonlinear solution if the homogeneous equations associated with (2.25) subject to the conditions (2.26) admit no other eigensolutions except ϕ_{10}.

2.3. *Method of solution*

We notice that ϕ_{nm} is related to the $O(A^{n+2m})$ terms of the complete solution ψ, and solve the set of equations in the sequence of ascending values of $l = n+2m$. For $l = 0$ we have the basic flow (2.5) and $\phi_{00} = 0$. For $l = 1$ we must solve the Orr–Sommerfeld problem

$$\{L_1 - \lambda_0 M_1\}\phi_{10} = 0, \quad \phi'_{10} = \phi_{10} = 0 \quad \text{at} \quad y = \pm 1, \tag{2.28}$$

with $\phi_{10}(0) = 1$ for given values of R and α, where the prime denotes d/dy. This provides the principal eigenvalue $\lambda_0 = a_0 - i\omega_0$ and the eigenfunction ϕ_{10} for the solution (2.10) of the linear stability problem. Following Stuart (1960), we also introduce the adjoint eigenfunction $\Phi(y)$ related to λ_0 that satisfies the system adjoint to (2.28). From orthogonality relations (Eckhaus 1965) we obtain

$$\int_{-1}^{1} \tilde{\Phi}\{L_1 \phi_{10} dy = c, \quad c \neq 0, \tag{2.29}$$

where the value of c depends on the normalization of Φ. For any sufficiently smooth function χ that satisfies $\chi' = \chi = 0$ at $y = \pm 1$, it can be shown (Reynolds & Potter 1967*b*) that

$$\int_{-1}^{1} \tilde{\Phi}\{L_1 - \lambda_0 M_1\} \chi \, dy = 0. \tag{2.30}$$

Before we turn to the next step, $l = 2$, of evaluating ϕ_{01} and ϕ_{20}, we consider the general problem of solving (2.25) for $n = 0$, $m > 0$. Integrating once the equations for ϕ_{0m} over y, introducing $u_{0m} = d\phi_{0m}/dy$, and applying the condition that the mean-pressure gradient should be constant, we obtain after some rearrangement

$$\left\{\frac{d^2}{dy^2} - 2ma_0 R\right\} u_{0m} = 2R \sum_{\mu=1}^{m-1} (m-\mu)\, a_\mu u_{0m-\mu} + G_{0m}, \tag{2.31a}$$

$$G_{0m} = \sum_{\nu=1}^{m} \sum_{\mu=0}^{m-\nu} i\nu\alpha R \frac{d}{dy} \{\phi_{-\nu m}\phi'_{\nu m-\nu-\mu} - \phi_{\nu m-\nu-\mu}\phi'_{-\nu\mu}\}, \tag{2.31b}$$

$$u_{0m} = 0 \quad \text{at} \quad y = \pm 1. \tag{2.31c}$$

As has been pointed out by Davey & Nguyen (1971) in a similar context, the associated homogeneous problem

$$\left\{\frac{d^2}{dy^2} - 2ma_0 R\right\} f = 0, \quad f = 0 \quad \text{at} \quad y = \pm 1, \tag{2.32}$$

admits eigensolutions when $a_0 = -(k\pi)^2/8mR$, $k = 1, 2, \ldots$. The occurrence of such free modes violates our assumption that the solution is purely forced by the fundamental, and invalidates the expansion in terms of a single amplitude. Even if the eigenvalues are not exactly met, ill-conditioned problems arise from solving (2.31) in their neighbourhood. Therefore our expansion procedure can be expected to provide rational approximations only if $a_0 \geqslant 0$, i.e. for neutral or unstable normal modes. This restricts the applicability to the range of wavenumbers $\alpha_1 \leqslant \alpha \leqslant \alpha_2$, where $\alpha_1(R)$ and $\alpha_2(R)$ are the points at the lower and upper branch respectively of the neutral curve for $R > R_c$ and $\alpha_1 = \alpha_2 = \alpha_c$ at $R = R_c$. In this range, the evaluation of u_{0m} (and hence ϕ_{0m}) is a straightforward matter, since the right-hand side of (2.31) is a known function of y at the stage when this equation is to be solved. In particular, the evaluation of u_{01} and ϕ_{01} poses no problem.

For $a_0 \geqslant 0$ it is unlikely that the expansion procedure will fail owing to eigensolutions of the homogeneous problems

$$\{L_n - (2ma_0 + n\lambda_0) M_n\} f = 0, \quad f' = f = 0 \quad \text{at} \quad y = \pm 1, \tag{2.33}$$

associated with (2.25) for the harmonic contributions ϕ_{nm}, $n > 0$. Since eigenvalues with a real part $(2m+n)a_0 > 0$ do not occur for $n\alpha > \alpha_2(R)$, the system (2.32), $n > 1$, admits no eigensolutions for the range of Reynolds numbers as high as $R = 10^5$. Disregarding this, it would be an unlikely contingency† if real and imaginary parts of $2ma_0 + n\lambda_0$ agreed with the principal eigenvalue for R, $n\alpha$. This situation would be easier to meet with $a_0 < 0$ owing to the infinite sequence of decaying eigenmodes. For $n = 1$, $m > 0$ and $a_0 > 0$ the system (2.33) admits no eigensolutions, since, by definition of the principal eigenvalue, there is no eigenvalue with an amplification rate $(2m+1)a_0 > a_0$. The case $a_0 = 0$ does not invalidate the expansion, as we shall see.

With this background, the evaluation of ϕ_{20} and ϕ_{30} in the next step is a straightforward matter. For $l = 3$ we are also faced with determining the distortion ϕ_{11} of the fundamental together with the complex Landau constant λ_1 from the equations

$$\{L_1 - (2a_0 + \lambda_0) M_1\} \phi_{11} = \lambda_1 M_1 \phi_{10} + F_{11}, \quad \phi'_{11} = \phi_{11} = 0 \quad \text{at} \quad y = \pm 1, \tag{2.34a, b}$$

$$F_{11}(y) = N[\phi_{01}, \phi_{10}] + N[\phi_{10}, \phi_{01}] + N[\phi_{-10}, \phi_{20}] + N[\phi_{20}, \phi_{-10}], \tag{2.34c}$$

† Note that this coincidence can be detected during the numerical evaluation.

with the additional condition (2.27) for $m = 1$:

$$\phi_{11}(y_0) = 0, \quad y_0 = 0. \tag{2.35}$$

We consider first the case $a_0 = 0$, which leads to

$$\{L_1 - \lambda_0 M_1\}\phi_{11} = \lambda_1 M_1 \phi_{10} + F_{11}, \quad \phi'_{11} = \phi_{11} = 0 \quad \text{at} \quad y = \pm 1. \tag{2.36}$$

The associated homogeneous problem is solvable and identical with the Orr–Sommerfeld problem (2.28). Moreover, ϕ_{11} satisfies the boundary conditions required for the function χ in (2.30). By multiplying (2.36) with the adjoint eigenfunction Φ, integrating from $y = -1$ to $y = 1$, and applying (2.30) for the left-hand side, we obtain

$$0 = \int_{-1}^{1} \Phi(\lambda_1 M_1 \phi_{10} + F_{11})\, dy. \tag{2.37}$$

Hence the value of the Landau constant λ_1 is given by

$$\lambda_1 = -\frac{1}{c}\int_{-1}^{1} \Phi F_{11}\, dy \quad \text{for} \quad a_0 = 0, \tag{2.38}$$

where c is the non-zero constant defined by (2.29). This is the classical way to determine the Landau constant from the orthogonality condition. With known λ_1, the solution of (2.36) can be found to within an arbitrary multiple of the solution ϕ_{10} of the homogeneous problem (2.28). Since $\phi_{10}(y_0) = 1$ for $y_0 = 0$, a unique solution $\phi_{11}(y)$ is provided by condition (2.35).

For $a_0 \neq 0$, (2.34) could be solved for arbitrary values of λ_1, as observed by Watson (1960). However, with the condition (2.35) a unique solution is selected and λ_1 uniquely determined. In order to show this, we write $\phi_{11} = \lambda_1 \chi_0 + \chi_1$, and obtain from (2.34)

$$\{L_1 - (2a_0 + \lambda_0) M_1\}\chi_0 = M_1 \phi_{10}, \quad \chi'_0 = \chi_0 = 0 \quad \text{at} \quad y = \pm 1, \tag{2.39}$$

$$\{L_1 - (2a_0 + \lambda_0) M_1\}\chi_1 = F_{11}, \quad \chi'_1 = \chi_1 = 0 \quad \text{at} \quad y = \pm 1. \tag{2.40}$$

Both of these problems are uniquely solvable, and it is easy to verify that $-2a_0\chi_0 = \phi_{10}$. Thus

$$\phi_{11}(y) = \chi_1(y) - \frac{\lambda_1}{2a_0}\phi_{10}(y) \quad \text{for} \quad a_0 \neq 0. \tag{2.41}$$

In order to satisfy (2.35) with $\phi_{10}(y_0) = 1$ according to (2.27), the constant λ_1 must be

$$\lambda_1 = 2a_0 \chi_1(y_0), \quad y_0 = 0 \quad \text{for} \quad a_0 \neq 0. \tag{2.42}$$

Substitution into (2.41) provides

$$\phi_{11}(y) = \chi_1(y) - \chi_1(y_0)\,\phi_{10}(y). \tag{2.43}$$

In this way, we obtained two different definitions of the Landau constant λ_1. We note that the derivation of (2.42) neither exploits the orthogonality condition (2.29) nor requires knowledge of the adjoint eigenfunction. A dissenting statement in Davey (1978) was subsequently revised by Davey (private communication). It remains to be shown that $\lambda_1 = \lambda_1(a_0)$ defined by (2.42) tends to the value $\lambda_1 = \lambda_1^*$ according to (2·38) as $a_0 \to 0$.

For small $a_0 \neq 0$ we follow Watson (1960) and expand $\chi_1(y)$ in powers of $2a_0$. From (2.41) and (2.35) it is obvious that $\chi_1 = O(a_0^{-1})$ as $a_0 \to 0$. Therefore we set

$$\chi_1 = \frac{1}{2\alpha_0}\chi_1^{-1} + \chi_1^0 + 2a_0\chi_1^1 + \dots, \tag{2.44}$$

and obtain from (2.40) by comparison of like powers of $2a_0$ the equations

$$\{L_1 - \lambda_0 M_1\}\chi_1^{-1} = 0, \tag{2.45}$$

$$\{L_1 - \lambda_0 M_1\}\chi_1^0 = M_1\chi_1^{-1} + F_{11}, \ldots. \tag{2.46}$$

The functions χ_1^{-1}, χ_1^0, ... satisfy separately the same boundary condition as χ_1. The solution of (2.45) is $\chi_1^{-1} = \Lambda_1\phi_{10}$, with an as-yet arbitrary constant Λ_1. By multiplying (2.46) with Φ and integrating from $y = -1$ to $y = 1$, we obtain with (2.29) and (2.30)

$$\Lambda_1 = -\frac{1}{c}\int_{-1}^{1} \Phi F_{11}\,dy \quad \text{for} \quad a_0 \neq 0, \quad a_0 \text{ small}. \tag{2.47}$$

This is formally identical with (2.38), but note that c, Φ and F_{11} in (2.47) may differ by terms $O(a_0)$ from the values at the neutral curve taken in (2.38). Therefore

$$\Lambda_1 = \lambda_1^* + O(a_0) \quad \text{as} \quad a_0 \to 0. \tag{2.48}$$

In order to satisfy (2.35) we substitute (2.44) into (2.41) and obtain

$$\phi_{11}(y_0) = \frac{1}{2a_0}[(\Lambda_1 - \lambda_1)\phi_{10}(y_0) + 2a_0\chi_1^0(y_0) + \ldots] = 0, \tag{2.49}$$

hence

$$\lambda_1 = \Lambda_1 + O(a_0) = \lambda_1^* + O(a_0) \quad \text{as} \quad a_0 \to 0. \tag{2.50}$$

These results show that the constant λ_1 as well as the function ϕ_{11} pass smoothly through neutral points.

With the determination of λ_1 and ϕ_{1m} from (2.34), (2.35), the step $l = 3$ is complete. At closer analysis, the differential equations encountered at higher order are of either one of the types already discussed in the previous steps. In particular, we find for $n = 1$, $m > 1$ the equations

$$\{L_1 - (2ma_0 + \lambda_0)M_1\}\phi_{1m} = \lambda_m M_1\phi_{10} + H_{1m}, \tag{2.51a}$$

$$H_{1m} = \sum_{\mu=1}^{m-1}[2(m-\mu)a_\mu + \lambda_\mu]M_1\phi_{1m-\mu} + F_{1m}, \tag{2.51b}$$

with the usual boundary conditions and known functions $H_{1m}(y)$. With the same arguments as for $m = 1$ we obtain now for points at the neutral curve

$$\lambda_m = -\frac{1}{c}\int_{-1}^{1} \Phi H_{1m}\,dy \quad \text{for} \quad a_0 = 0, \quad m > 0, \tag{2.52}$$

which contains (2.38) since $H_{11} = F_{11}$. With these values of λ_m and the additional conditions (2.27) the equations (2.51) can be uniquely solved for ϕ_{1m}, $m > 0$, as $a_0 = 0$. Otherwise we proceed as for $m = 1$ and obtain instead of (2.42) and (2.43) the solution

$$\lambda_m = 2ma_0\chi_m(y_0), \quad y_0 = 0 \quad \text{for} \quad a_0 \neq 0, \quad m > 0, \tag{2.53}$$

$$\phi_{1m} = \chi_m(y) - \chi_m(y_0)\phi_{10}(y), \tag{2.54}$$

where in analogy to (2.40)

$$\{L_1 - (2ma_0 + \lambda_0)M_1\}\chi_m = H_{1m}, \quad \chi_m' = \chi_m = 0 \quad \text{at} \quad y = \pm 1. \tag{2.55}$$

In this way, the Landau constants λ_m and functions $\phi_{nm}(y)$ are uniquely determined up to arbitrary order.

Hence our expansion in the time-dependent amplitude $A(t)$ provides a rational approximation method for studying the nonlinear growth of disturbances in a large range of Reynolds numbers. The assumption that the nonlinear solution is completely determined by the fundamental component is satisfied for $a_0 \geqslant 0$ as long as, at given R and α, the harmonic points R and $n\alpha$, $n > 1$, are in the stable domain. For plane Poiseuille flow this condition is always satisfied for Reynolds numbers as high as $R = 10^5$. The perturbation series provides the solution (2.12) of the nonlinear problem (2.14) and (2.15) for amplitudes $A(t)$ within the convergence domain of the perturbation series. It also contributes to the determination of equilibrium amplitudes A_e within this domain.†

We suggest that the invalidity of the expansion about the basic flow for points R, α in the stable domain has its origin in the ill-posedness of the problem under consideration. Such an expansion would be equivalent to an attempt to trace the history back to the non-unique initial conditions, which led asymptotically to the small neighbourhood of the basic flow. The non-uniqueness manifests itself in the existence of numerable sets of decaying eigenfunctions for R, $n\alpha$, $n \geqslant 0$. On the contrary, the existence of only a single growing mode guarantees that the nonlinear solution emerges from the neighbourhood of the basic flow in a well-defined way.

A final remark on an alternative definition of the amplitude seems to be in order. Starting from the L_2 norm but avoiding nonlinearity in ψ_1, the size of the fundamental can be measured by

$$A(t) = \int_{-1}^{1} \tilde{\phi}_{10}(y)\, \psi_1(y,t)\, dy, \tag{2.56}$$

and one obtains from (2.16)

$$\int_{-1}^{1} \tilde{\phi}_{10}(y)\, \phi_1(y,t)\, dy = 1. \tag{2.57}$$

Introducing (2.23) for ϕ_1 and comparing in powers of A^2 provides

$$\int_{-1}^{1} |\phi_{10}|^2\, dy = 1, \quad \int_{-1}^{1} \tilde{\phi}_{10}\phi_{1m}\, dy = 0, \quad m > 0, \tag{2.58}$$

leaving the phase of ϕ_{10} yet to be fixed. Hence, with the definition (2.56) of the amplitude, the functions ϕ_{1m} must be orthogonal to the eigenfunction ϕ_{10} of the linear problem. It is a straightforward matter to adapt our method of solution to the modified conditions and to verify that the Landau constants λ_m and functions ϕ_{nm} are still uniquely determined. We note, however, that changing the definition of the amplitude may well affect the range of validity of the series expansion.

3. Comparison with previous work

The expansion formalism in §2 is distinguished from previous expansions in $A(t)$ by the definition of the amplitude and, as consequences, the uniqueness in determining the Landau constants λ_m and functions $\phi_{1m}(y)$ for $a_0 \geqslant 0$ and a new way of determining the constants λ_m for $a_0 > 0$.

Most of the previous work on expansions in the time-dependent amplitude, including that of Stuart (1960) and Watson (1960), based the definition of the amplitude solely on the asymptotic behaviour of the fundamental:

$$\psi_1(y,t) \to A(t)\, \phi_{10}(y) \quad \text{as} \quad A \to 0. \tag{3.1}$$

† A successful application to equilibrium states of Taylor vortices in circular Couette flow has been recently reported (Herbert 1981).

By comparison with (2.16) and (2.23) for $n = 1$, this definition can be written as

$$\psi_1(y,t) = A(t)\,(\phi_{10}(y) + O(A^2)), \tag{3.2}$$

with ϕ_{10} normalized according to (2.11). The definition (3.2) of $A(t)$ agrees with (2.16) and (2.18) in the leading term, but differs by neglecting higher-order terms. With (3.2), a change in the size of the fundamental can be either accounted for by a change in $A(t)$ or by a contribution that is concealed in the $O(A^2)$ terms. As a consequence, the conditions (2.27) on the $\phi_{1m}, m > 0$, are lost, which are essential for the construction of a unique solution. We fully confirm Eagles' (1971) conclusion that the 'lack of uniqueness in the representation of the physical system arises from a certain lack of precision in the definition' of the amplitude according to (3.2). We also note that Landau's amplitude equation

$$a = \frac{1}{A}\frac{dA}{dt} = \sum_{m=0}^{\infty} a_m A^{2m} \tag{3.3}$$

cannot be considered as a key equation for the discussion of nonlinear disturbance behaviour if changes in the disturbance size are hidden in the functions ϕ_{nm}.

The theory given by Stuart (1960) concentrated only on the leading terms of the expansion in the neighbourhood of the critical point R_c, α_c in order to decide on subcritical instability or supercritical stability of the basic flow. Stuart explicitly states that 'no attempt has been made to develop a perturbation series'. Under these conditions it is sufficient to calculate λ_1 for $a_0 = 0$ from (2.38) and to show by (2.48) that the first Landau constant is a continuous function of a_0 in the vicinity of $a_0 = 0$. The derivation of (2.38) and (2.48) makes no use of (2.35), and therefore the weak definition (3.2) of the amplitude is sufficient to achieve the goals of Stuart's analysis.

The deficiency of the amplitude definition turned up in Watson's (1960) attempt to develop a valid perturbation expansion of the complete solution of the Navier–Stokes equations. Although it is not clear whether his attempt was from the outset restricted to the vicinity of the neutral curve, the Landau constants λ_m can be arbitrarily chosen unless $m|a_0|$ is sufficiently small. In fact, with the conditions (2.27) for $m > 0$ missing, (2.34) and (2.51) can be solved for arbitrary λ_m if $a_0 > 0$. It is obvious from equations like (2.41) that the choice of a particular value for λ_m fixes the contribution of $\phi_{10}(y)$ to $\phi_{1m}(y)$, and hence the value of $\phi_{1m}(y_0)$ at the channel centre. This indicates clearly how the information on the size of the fundamental can be redistributed between the amplitude equation (3.3) and the function $\phi_1(y_0, t)$.

When $a_0 \to 0$ the problem of solving (2.34) and (2.51) is ill-conditioned. For $a_0 = 0$, the constant λ_1 is uniquely determined by the orthogonality condition (2.38). Formally similar conditions (2.52) for λ_m, $m > 1$, can be derived, but the ϕ_{1m}, $m > 0$, can be found only to within an arbitrary multiple of ϕ_{10}. Therefore the λ_m for $m > 1$ are non-unique. Watson remarks that this arbitrariness corresponds to the arbitrariness with which the series may be arranged. The arrangement of the infinite series may be crucial for the convergence properties, but should be irrelevant for representing the physical solution uniquely. In applications, however, only a finite number of terms is known and the rearrangement is impracticable. Hence, a unique physical solution cannot be obtained at any finite order of truncation.

In the neighbourhood $m|a_0| \ll 1$ of the neutral curve, Watson introduced an expansion of the ϕ_{1m} in the small amplification rate a_0 and extended the range of applicability of the orthogonality condition by reasons of steadiness. His definition of the first Landau constant for $a_0 \neq 0$ agrees with Λ_1 given by (2.47). We have shown in (2.50) that Λ_1 differs from the exact value λ_1 by terms $O(a_0)$. Thus the Landau constant based on the orthogonality condition becomes increasingly inaccurate with increasing amplification rate. Since the order estimate conceals any quantitative information on the deviation $\Lambda_1 - \lambda_1$, the results of Watson's method for $a_0 \neq 0$ cannot be considered fully conclusive. This applies, for instance, to the results of Pekeris & Shkoller (1967) for plane Poiseuille flow or to those of Eagles (1971) for the Taylor–Couette problem. In particular, the secondary instability studied by Eagles occurs at definite supercritical Reynolds numbers, with values of a_0 greater than unity. We note that the inaccuracy of Λ_1 is not directly related to the weakness of the amplitude definition; it originates from using the orthogonality condition with (2.34) for $a_0 \neq 0$ although the associated homogeneous problem is not solvable. The invalidity of Watson's method for $a_0 < 0$ has been discovered by Davey & Nguyen (1971) in an attempt to utilize this method for a study on finite-disturbance behaviour in pipe flow, where always $a_0 < 0$. Itoh (1977*a*) concluded that the restriction of Watson's approach to the vicinity of the neutral curve originates from the assumption that the nonlinear solution is uniquely determined by the fundamental component. This holds true for $a_0 < 0$ but not for $a_0 > 0$.

Eckhaus' (1965) technique of expanding the nonlinear solution in terms of eigenfunctions of the linear problem with time-dependent coefficients is not directly comparable to our amplitude expansion. With only one coefficient taken into account, the first two terms of the Landau series (3.3) can be obtained as from Stuart's (1960) approach. The distortion of the fundamental as a function of y, however, requires introduction of additional eigenfunctions and coefficients. Instead of introducing higher-order terms, this technique provides an expanding coupled system of 'amplitude equations' for the coefficients.

A well-defined amplitude and an alternative formula for λ_1 similar to (2.42) had already been used by Itoh (1974) for spatially growing disturbances. However, the consequences for the formulation of a rational expansion procedure were concealed by the low order of truncation and not pursued any further. In a more recent analysis of formal expansion procedures, Itoh (1977*a*) raised serious objections on the grounds that 'the asymptotic theory with the disturbance amplitude as the small parameter turns out not to provide an estimate of finite equilibrium amplitudes' and accordingly misses one of its major goals. However, this conclusion from his equation (3.5) is unjustified. Itoh observes that for $a_0 \neq 0$ the ratio A_e/ϵ (his $|a_1^{(0)}|$) tends to infinity as $\epsilon \to 0$ in contradiction to his assumption that $A_e/\epsilon = O(1)$. This is fully consistent with the fact that the equilibrium amplitude A_e cannot be zero except at a bifurcation point, $a_0 = 0$, and fails to prove his statement.

Itoh (1977*a*) suggested an asymptotic theory for small disturbances that rests on Eckhaus' eigenfunction expansions for all Fourier components and estimates analogous to (2.20). He retained only the most important terms and obtained the first Landau constant by considering trajectories in phase space. These considerations led him to cancel the terms multiplied by a_0 in the equations for mean flow and harmonics, resulting in a set of equations previously used by Ellingsen *et al.* (1970). Itoh did not explicitly restrict his formulation to equilibrium states, but we will show in §4 that his equations are identical with the low-order equations of a special method of false problems. These methods provide physical solutions only for equilibrium states, but non-physical solutions for $A \neq A_e$.

Eagles (1971) apparently made the only successful attempt to include the second Landau constant a_2 in his analysis of finite-amplitude Taylor vortices. He achieved by rational arguments an additional condition similar to (2.27) for $m = 1$ which uniquely determines a_2 and incorporates the growth of the disturbance into the amplitude equation. Since the implications on the determination of the constants for $a_0 > 0$ were not revealed by his arguments, he calculated the Landau constants from the orthogonality condition. Hence the results are asymptotically valid as $a_0 \to 0$, but subject to reservations at supercritical Reynolds numbers with values of a_0 greater than unity.

4. Methods of false problems

A method of false problems was first suggested by Reynolds & Potter (1967*b*) for a direct attack on equilibrium states in plane Poiseuille flow. A rational extension of this method was obtained by Herbert (1978*b*, 1980) with a modified amplitude definition analogous to (2.16) and (2.18). Without particularly advocating the use of these methods, we generalize in the following the method of Reynolds & Potter into a class of basically equivalent formulations, which are distinguished by their ranges of validity and convergence.

For equilibrium states, the amplitude A_e does not change with time, and therefore the nonlinear amplification rate vanishes: $a(A_e) = 0$. According to (2.24) we obtain

$$\frac{1}{A}\frac{dA}{dt} = a = \sum_{m=0}^{\infty} a_m A^{2m} = 0 \quad \text{for} \quad A = A_e. \tag{4.1}$$

Applying this to the series (2.23) for $\phi_n(y,t)$, it is easy to see that the term $\partial(M_n \phi_n)/\partial t$ in (2.21) vanishes, and (2.25) can be replaced by

$$\{L_n - n\lambda_0 M_n\}\phi_{nm} = \sum_{\mu=1}^{m} n\lambda_\mu M_n \phi_{nm-\mu} + F_{nm}, \tag{4.2}$$

with $F_{nm}(y)$ as defined in (2.25*b*). The boundary conditions (2.26) and the conditions (2.27) remain unchanged. We retain for convenience the notation of §2, although the functions ϕ_{nm}, F_{nm} and constants λ_m in (4.2) will not be identical with those in (2.25) except for $l < 2$. Obviously, the results from solving (4.2) provide a solution of the physical problem only if $A = A_e$ and the relation (4.1) is satisfied. For $A \neq A_e$ we obtain solutions of physically meaningless 'false problems', which are to be considered as a formal means for constructing the equilibrium solution.

As before, the set of equations (4.2) is solved in the sequence of ascending values of $l = n + 2m$. The results for $l < 2$ agree with those of §2. However, there are essential changes in the various types of equations for $n = 0$, $n = 1$ and $n > 1$. For the mean-flow distortions u_{0m}, $m > 0$, we obtain instead of (2·31)

$$u''_{0m} = G_{0m}, \quad u_{0m} = 0 \quad \text{at} \quad y = \pm 1. \tag{4.3}$$

These equations admit no eigensolutions of the associated homogeneous problems and provide a unique solution for arbitrary values of a_0. Thus the method of Reynolds & Potter is not invalidated by free mean-flow modes. Therefore it is not only valid in the unstable but also in the stable domain, if no eigensolutions of the harmonic equations occur.

For the distortions ϕ_{1m} of the fundamental, we obtain from (4.2) with $n = 1$ throughout equations of the type (2.36), with the associated homogeneous problem solvable. Hence for arbitrary R, α and related a_0 the Landau constants λ_m, $m > 0$, are determined by the orthogonality condition

$$\lambda_m = -\frac{1}{c}\int_{-1}^{1} \Phi H_{1m}\, dy, \quad H_{1m} = \sum_{\mu=1}^{m-1} \lambda_\mu M_1 \phi_{1m-\mu} + F_{1m}. \tag{4.4}$$

The functions ϕ_{1m} are uniquely determined owing to the conditions (2.27). For $n > 1$, the harmonic equations provide unique solutions ϕ_{nm}, if $n\lambda_0$ is neither identical nor very close to one of the eigenvalues of the Orr–Sommerfeld problem for R, $n\alpha$. Otherwise the problem would be singular or ill-conditioned. As already mentioned in §2.3, an invalidation of the expansion in a single amplitude by harmonic resonance is unlikely in the unstable domain, but may well occur in the stable domain. An example for this situation has been given by Itoh (1977*b*) in his study on pipe flow. We suggest that this problem can be overcome by choice of another method of false problems from an infinite variety of such methods.

Writing (2.21) with $\partial\phi_n/\partial t = 0$ in the form

$$\{L_n - (na - in\omega)\, M_n\}\,\phi_n = F_n, \tag{4.5}$$

and adding arbitrary multiples of the identity $a \equiv 0$ in the parentheses, leads to the more-general equations

$$\{L_n - (q_n a - in\omega)\, M_n\}\,\phi_n = F_n \tag{4.6}$$

for equilibrium states. This results in the equations

$$\{L_n - (q_n a_0 - in\omega_0)\, M_n\}\,\phi_{nm} = \sum_{\mu=1}^{m} (q_n a_\mu - in\omega_\mu)\, M_n \phi_{nm-\mu} + F_{nm} \tag{4.7}$$

for the ϕ_{nm}. There are some restrictions on the choice of the numbers q_n. In order to be compatible with the linear problem (2.21), we must set $q_1 = 1$. This guarantees at the same time the occurrence of λ_m in the right-hand side of (4.7) for $n = 1, m > 0$. In order to avoid eigensolutions of the mean-flow equations, $q_0 a_0 \geqslant 0$ must be satisfied. The choice $q_0 = 0$ avoids invalidation by mean-flow resonance for arbitrary a_0 and hence for arbitrary R, α. For flows without a neutral curve and $a_0 < 0$ throughout, $q_0 < 0$ would be another appropriate choice. With $q_n = n$ we obtain the straightforward extension of the method of Reynolds & Potter, whereas the method suggested by Itoh (1977*a*) for $n \leqslant 2$ is obtained by setting $q_0 = q_2 = 0$. With $q_n > 0$, $n > 1$, the danger of harmonic resonance persists for $a_0 < 0$. On the other hand, with $q_n < 0$, $n > 1$, this danger is completely removed only for subcritical Reynolds numbers and hence for flows without a neutral curve. For supercritical Reynolds numbers R and small wavenumbers $\alpha < \alpha_1(R)$, harmonic resonance may occur with $q_n a_0 > 0$ if $\alpha_1 < n\alpha < \alpha_2$ for some $n > 1$. Therefore $q_n = 0$ for $n > 1$ appears as the most-appropriate choice, albeit not mandatory.

The choice of a special set of q_n, $n \neq 1$, is equivalent to a rearrangement of the terms of the infinite series introduced into (4.6). In fact, the change of the multiple of $a = 0$ is equivalent with replacing some multiple of a_0 by a multiple of the higher-order terms $-\sum_{m=1}^{\infty} a_m A^{2m}$. Obviously, these manipulations require convergence of the series for some range of the amplitude A. We have as yet no criterion for specifying the arrangement with fastest convergence or largest convergence domain. However, the results should agree at sufficiently high approximation if these methods are used inside their respective convergence domain. This aspect has been clearly pointed out by Davey (1978) in his comparison of Reynolds & Potter's method ($q_2 = 2$) with Itoh's method ($q_2 = 0$) for pipe flow. In our view, Davey's results for different values of his parameter λ (which corresponds to our $q_2 = 2(1-\lambda)$) are all equivalent, but indicate that judicious results require higher-order terms to be taken into account. This could also contribute to estimates on the convergence domain for different sets of q_n, $n > 1$. The convergence problem for the methods of false problems is in practice more serious than for the method of §2. Whereas $A(t)$ can be restricted to small values within the convergence domain, the equilibrium amplitude A_e (if it exists) at given R, α has a definite value, which may be outside the radius of convergence.

In spite of this convergence problem, the methods of false problems are of some advantage. They are formally applicable for arbitrary R, α and all values of a_0. They are also computationally simpler than the method of §2, since the differential operator in (4.7) is independent of the index m of ϕ_{nm}. Moreover, they provide as yet the only means for attacking possible equilibrium states in flows without a neutral curve. The objections of Rosenblat & Davis (1979) derived from their study of model problems that bifurcate at infinite Reynolds number and at an eigenvalue of infinite multiplicity may not fully apply to using methods of false problems with single eigenvalues at finite Reynolds numbers. However, the choice of an appropriate fundamental mode is not obvious. Moreover, results can be only considered as conclusive if the series are evidently used inside their radii of convergence. For studies on equilibrium states in flows with a neutral curve, we suggest the application of parameter expansions as discussed in §1. These parameter expansions are superior to the methods of false problems in that they provide at once the equilibrium solutions in the full convergence domain.

REFERENCES

BUSSE, F. H. 1978 *Rep. Prog. Phys.* **41**, 1929.

CHEN, T. S. & JOSEPH, D. D. 1973 *J. Fluid Mech.* **58**, 337.

COFFEE, T. 1977 *J. Fluid Mech.* **83**, 401.

DAVEY, A. 1962 *J. Fluid Mech.* **14**, 336.

DAVEY, A. 1978 *J. Fluid Mech.* **86**, 695.

DAVEY, A. & NGUYEN, H. P. F. 1971 *J. Fluid Mech.* **45**, 701.

EAGLES, P. M. 1971 *J. Fluid Mech.* **49**, 529.

ECKHAUS, W. 1965 *Studies in Nonlinear Stability Theory*. Springer.

ELLINGSEN, T., GJEVIK, B. & PALM, E. 1970 *J. Fluid Mech.* **40**, 97.

FASEL, H. 1974 Untersuchungen zum Problem des Grenzschichtumschlages durch numerische Integration der Navier–Stokes–Gleichungen. Dissertation, Universität Stuttgart.

GAUNT, D. S. & GUTTMANN, A. J. 1974 In *Phase Transitions and Critical Phenomena* (ed. C. Domb & M. S. Green), vol. 3, p. 181. Academic.

GORKOV, L. P. 1957 *Sov. Phys. JETP* **6**, 311.

HERBERT, TH. 1977 *Laminar–Turbulent Transition. AGARD Conf. Proc.* no. 224, paper no. 3.

HERBERT, TH. 1978*a* Die neutrale Fläche der ebenen Poiseuille–Strömung. Habilitationsschrift, Universität Stuttgart.

HERBERT, TH. 1978*b* *A.I.A.A. Paper* no. 78–1125.

HERBERT, TH. 1980 *A.I.A.A. J.* **18**, 243.

HERBERT, TH. 1981 In *Seventh International Conference on Numerical Methods in Fluid Dynamics* (ed. W. C. Reynolds & R. W. MacCormack). Lecture Notes in Physics, vol. 141, p. 200. Springer.

ITOH, N. 1974 *Trans. Japan Soc. Aero. Space Sci.* **17**, 160.

ITOH, N. 1977*a* *J. Fluid Mech.* **82**, 455.

ITOH, N. 1977*b* *J. Fluid Mech.* **82**, 469.

JOSEPH, D. D. & SATTINGER, D. H. 1972 *Arch. Rat. Mech. Anal.* **45**, 79.

KIRCHGÄSSNER, K. & SORGER, P. 1969 *Q. J. Mech. Appl. Math.* **22**, 183.

KLEISER, L. 1982 Numerische Simulationen zum laminar-turbulenten Umschlagsprozeß der ebenen Poiseuille–Strömung. Dissertation, Universität Karlsruhe.

Kuo, H. L. 1961 *J. Fluid Mech.* **10**, 611.

Kuo, H. L. & Platzman, G. W. 1961 *Beitr. Phys. Atmos.* **33**, 137.

Malkus, W. V. R. & Veronis, G. 1958 *J. Fluid Mech.* **4**, 225.

Orszag, S. A. & Kells, L. C. 1980 *J. Fluid Mech.* **96**, 159.

Orszag, S. A. & Patera, A. T. 1981 In *Transition and Turbulence* (ed. R. E. Meyer), p. 127. Academic.

Palm, E. 1960 *J. Fluid Mech.* **8**, 183.

Pekeris, C. L. & Shkoller, B. 1967 *J. Fluid Mech.* **29**, 31.

Reynolds, W. C. & Potter, M. C. 1967*a* A finite-amplitude state-selection theory for Taylor-vortex flow. Unpublished Stanford Univ. Paper.

Reynolds, W. Ç. & Potter, M. C. 1967*b* *J. Fluid Mech.* **27**, 465.

Rosenblat, S. & Davis, S. H. 1979 *SIAM J. Appl. Math.* **37**, 1.

Shanks, D. 1955 *J. Math. & Phys.* **34**, 1.

Stewartson, K. 1975 *Fluid Dyn. Trans.* **7**, 101.

Stuart, J. T. 1960 *J. Fluid Mech.* **9**, 353.

Stuart, J. T. 1962 *Adv. Aero. Sci.* **3**, 121.

Stuart, J. T. 1971 *Ann. Rev. Fluid Mech.* **3**, 347.

Van Dyke, M. 1974 *Q. J. Mech. Appl. Math.* **27**, 423.

Van Dyke, M. 1975 *SIAM J. Appl. Math.* **28**, 720.

Van Dyke, M. 1978 *J. Fluid Mech.* **86**, 129.

Watson, J. 1960 *J. Fluid Mech.* **9**, 371.

Zahn, J. P., Toomre, J., Spiegel, E. A. & Gough, D. O. 1974 *J. Fluid Mech.* **64**, 319.